PITHA 97/40
gr-qc/9710032

Canonical Quantum Statistics of Schwarzschild Black Holes and Ising Droplet Nucleation

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Abstract

Recently it was shown[1] that the imaginary part of the canonical partition function of Schwarzschild black holes with an energy spectrum $E_n = \sigma\sqrt{n}\,E_P$, $n = 1, 2, \ldots$, has properties which - naively interpreted - leads to the expected unusual thermodynamical properties of such black holes (Hawking temperature, Bekenstein-Hawking entropy etc.).
The present paper interprets the same imaginary part in the framework of droplet nucleation theory in which the rate of transition from a metastable state to a stable one is proportional to the imaginary part of the canonical partition function. The conclusions concerning the emerging thermodynamics of black holes are essentially the same as before.
The partition function for black holes with the above spectrum was calculated exactly recently[1]. It is the same as that of the primitive Ising droplet model for nucleation in 1st-order phase transitions in 2 dimensions. Thus one might learn about the quantum statistics of black holes by studying that Ising model, the exact complex free energy of which is presented here for negative magnetic fields, too.

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1 Introduction

In a recent paper[1] I discussed properties of the canonical partition function for a quantum energy spectrum

$$E_n = \sigma\sqrt{n}E_P \ , n = 1, 2, \ldots, \quad E_P = c^2\sqrt{c\,\hbar/G} \ , \ \sigma = O(1) \ \ , \tag{1}$$

where the n-th level has the degeneracy

$$d_n = g^n \ , \ g > 1 \ \ . \tag{2}$$

The main interest in this spectrum comes from the many, differently justified, proposals (see the corresponding quotations in ref. [1]) that a quantized Schwarzschild black hole might have such a spectrum! The associated canonical partition function

$$Z(t, x) = \sum_{n=0}^{\infty} e^{nt} e^{-\sqrt{n}x} \ , \quad t = \ln g \ , \ x = \beta\sigma E_P \ \ , \tag{3}$$

converges only for $|g| \leq 1$, whereas one is interested in its properties for $g > 1$. However, the function $Z(g, x)$ can be continued analytically into the complex g-plane by means of the integral representation[1]

$$\begin{aligned} Z(g = e^t, x) &= \frac{x}{2\sqrt{\pi}} \int_1^{\infty} du \frac{e^{-x^2/(4\ln u)}}{\ln^{3/2} u} \frac{1}{u - g} && (4) \\ &= \frac{x}{2\sqrt{\pi}} \int_0^{\infty} \frac{d\tau}{\tau^{3/2}} e^{-x^2/(4\tau)} \frac{1}{1 - e^{(t-\tau)}} \ , && (5) \end{aligned}$$

which exhibits a branch cut of $Z(g, x)$ from $g = 1$ to $g = +\infty$. Approaching the cut from above yields a complex Z, with an imaginary part

$$Z_i(t, x) = \frac{\sqrt{\pi}x}{2t^{3/2}} e^{-x^2/(4t)} \tag{6}$$

and the principal value integral

$$Z_r(t, x) = \text{p.v.}\frac{1}{\pi} \int_0^{\infty} d\tau \, Z_i(\tau, x) \frac{1}{1 - e^{t-\tau}} \tag{7}$$

as its real part.
Amazingly one gets essentially all the expected thermodynamical properties of a black hole, including the Hawking temperature, if one - naively - uses the imaginary part Z_i above for their derivation. This is strange and provocative and needs further analysis. If such an approach can be substantiated, one will have an indication which quantum states of a black hole could be responsible for its unusual thermodynamical properties!
In the following I shall try to put the properties mentioned into a certain perspective and especially discuss their relationship to other approaches (which was not done in ref.[1]), namely the euclidean path integral (semiclassical) quantisation of black holes by Hawking and others and the theory of droplet nucleation associated with metastable states.
The paper is organized as follows: In chapter 2 the properties of the model are briefly summerized and its theoretical and physical potentials and limitations sketched. Parts of it recall well-known material which serves to have the discussion more or less self-contained. Chapter 3 discusses the relationship of the model to Hawking's euclidean path integral approach to the thermodynamics of black holes, where he also gets a purely imaginary partition function[2]! Chapter 4 deals with the nucleation of droplets occuring when a system changes from a metastable state to a stable one, e.g. in 1st-order phase transitions or in the decay of "false" ground states. Langer[3] was the first one to relate the imaginary part of the canonical partition function to the transition rate of such processes. Later Gross, Perry and Yaffe[4] interpreted the imaginary part of the semiclassical euclidean path integral - i.e. the canonical partition function - for the Schwarzschild black hole ("instanton") in this context, thus giving a different interpretation of the same quantity Hawking employed.
In chapter 5 the relation of the partition function (3) to that of the primitive Ising droplet model for nucleation in 2 dimensions is discussed and it is very easy to see that both are exactly the same. The integral representation (4) therefore provides an exact free energy expression for this model, too, which as far as I know, is new[5]. Chapter 6 presents some conclusions.

2 Scope of the model

In the background of trying to push the quantum analysis - including its quantum statistics - of the simple isolated Schwarzschild gravitational system as far as possible are the following considerations: The quantized radiation emitted by a "classical" Schwarzschild black hole has a thermal (Planck) distribution governed by the (Hawking) temperature[6]

$$k_B T_H \equiv \beta_H^{-1} = \frac{c^3 \hbar}{8\pi G M} \, , \tag{8}$$

where M is the mass of the black hole. Assuming that this temperature of the radiation arises from its thermal contact with the black hole of the same temperature, the immediate question was - and still is -, how does the black hole gets this temperature? As the temperature (8) is proportional to Planck's constant the quantum theory of gravity has to play a decisive role in its understanding. Especially one would like to identify the microscopic quantum gravitational degrees of freedom which yield the corresponding quantum statistics and its macroscopic thermodynamics.
Many attempts have been made to achieve this and I shall mention only a few of them in the following. The most active and enthusiastic reseach in this area presently takes place in connection with string theory. I shall say nothing about that in the following and refer to the very recent review by Horowitz[7].
In view of the lack of a generally accepted quantum theory of Einstein's General Relativity it seems difficult to extract those microscopic gravitational degrees of freedom within this framework. However, the situation is not quite es hopeless as it might appear. The reason is that Einstein's theory for a rotationally symmetric isolated gravitational system, Schwarzschild gravity, can consistently be quantized, namely by first identifying its classical observables in the sense of Dirac by solving the constraints associated with the gauge (diffeomorphims) degrees of freedom and then quantizing the remaining gauge-independent physical degrees of freedom afterwards. This has been done by Thiemann and myself[8] in the framework of Ashtekar's formalism and briefly afterwards by Kuchař[9] in the geometrodynamical framework, the results being equivalent. They have been briefly summerized in ref.[10].
The essential point is that this system classically has just one canonical pair

of (Dirac) observables, namely its mass M and as the canonically conjugate quantity of M a time functional $T[g;\Sigma]$ (of the metric g) which describes the difference $\tau_+ - \tau_-$ in proper time τ of two observers at the two (asymptotic) ends of the 1-dimensional spacelike hypersurface Σ which, together with the 2-dimensional spheres S^2, provides a time slicing of the 4-dimensional manifold. Formally the quantity

$$\delta = T[g;\Sigma] - (\tau_+ - \tau_-) \tag{9}$$

is the constant Dirac observable.

Thus, for an observer at the asymptotic end where $r \to +\infty$ on the Schwarzschild manifold with the time $\tau \equiv \tau_+$ there are - in the framework of this model - only two quantities available in order to describe the system: its Mass M and his proper time τ! Everything else has to be expressed by them (and some fundamental constants like $G, c, \hbar, k_B$). Examples are: the Schwarzschild radius $R_S = 2MG/c^2$ or any multiple thereof, the area $A = 4\pi R_S^2$ of the horizon, but also any time interval Δ associated with the system, i.e. $\Delta = \gamma R_S/c$, where γ is some number, e.g. $\gamma = 4\pi$ for the time period Δ_H of the euclidean section[2] of the complex Schwarzschild manifold.

Having eliminated the (infinite) gauge degrees of freedom classically one can now quantise the remaining physical degrees of freedom[8-10]. The extremely simple Schrödinger equation for this system,

$$i\hbar\partial_\tau\phi(\tau) = Mc^2\phi(\tau) \ , \tag{10}$$

has the plane wave solutions

$$\phi(M,\tau) = \chi(M)e^{-\frac{i}{\hbar}Mc^2\tau} \ , \tag{11}$$

where M is continuous and > 0. So at first sight there is no such spectrum like (1). However, one can get it by changing the boundary conditions.

My doing so in ref.[10] was stimulated by the paper of Bekenstein and Mukhanov[11], in which they discussed properties of the spectrum (1) with $\sigma^2 = \ln g/(4\pi), g = 2$. Already in 1974 Bekenstein[12] discussed this spectrum as arising from a Bohr-Sommerfeld type quantisation of the area of the horizon ($A \propto n\hbar$). Since then many authors have proposed such a spectrum (see refs. [3-23] in ref.[1]).

One knows from elementary quantum mechanics how to make the continuous momentum in plane waves discontinuous by periodic boundary conditions in space. Applying corresponding boundary conditions in time (with period Δ) to the plane waves (11) yields

$$c^2 M\Delta = 2\pi\hbar\, n \ , \ n = 1, 2, \ldots \ . \tag{12}$$

Introducing such boundary conditions is easier than justifying them by physical arguments. In any case it means that the system is represented by the plane waves only for the time Δ after which it is abruptly terminated, e.g., by forming of the horizon. This may happen, e.g., because the state described by the plane wave is a metastable one - see below. There are several physcal time scales $\Delta = \gamma R_S/c$ associated with a black hole all of which[10] have $\gamma \sim O(1)$, so that

$$E_n = \sqrt{\frac{\pi}{\gamma}}\,\sqrt{n}\,E_P \equiv \sigma\sqrt{n}\,E_P \ \ . \tag{13}$$

3 Euclidean path integral as canonical partition function of a Schwarzschild black hole

First I want to recall how Hawking's path integral approach to the canonical partition function in a semiclassical approximation leads to a purely imaginary partition function, too:

Shortly after Hawking's discovery that the quanta emitted by a collapsing black hole have a thermal distribution governed by the temperature (8) Gibbons and Hawking[13] employed the relationship between the canonical partition function of a system and the corresponding euclidean path integral in order to tackle the canonical quantum statistics of a black hole itself. For pure gravity that partition function Z is formally given by

$$Z = \int D[g] e^{-I_E[g]/\hbar} \ , \ \ I_E = \frac{1}{16\pi G}\int d^4x R(g)(g)^{1/2} + \text{surface terms}, \tag{14}$$

where I_E is the euclidean action of pure gravity (for more details of the path integral approach to quantum gravity see the reprint collection in ref.[2]). In principle one has to take into account all field configurations $g_{\mu\nu}$ which are periodic in euclidean time with period $\hbar\beta$ and which obey appropriate spatial

boundary conditions. In practice the path integral has been evaluated only semiclassically: One puts

$$g_{\mu\nu} = \overset{0}{g}_{\mu\nu} + h_{\mu\nu} \ , \tag{15}$$

where $\overset{0}{g}_{\mu\nu}$ is a classical solution of the euclidean field equations compatible with the boundary conditions just mentioned and the $h_{\mu\nu}$ denote (small) quantum fluctuations around the classical solution. Inserting (15) into the action (14) and keeping only terms up to order $O(h_{\mu\nu}^2)$ gives the semiclassical approximation

$$I_E^{scl} = I_E^{cl}[\overset{0}{g}] + I_E^{(2)} \ , \quad I_E^{(2)} = \int d^4x (\overset{0}{g})^{1/2} (hA(\overset{0}{g})h) \ , \tag{16}$$

where $A(\overset{0}{g})$ is a second order differential operator depending on the classical metric $\overset{0}{g}_{\mu\nu}$.

Gibbons and Hawking evaluated I_E^{cl} for the euclidean Schwarzschild metric and obtained

$$I_E^{cl}[\text{Schwarzschild}] = \frac{1}{2}\beta_H M c^2 \ , \tag{17}$$

where the period $\hbar\beta_H$ follows from the requirement that the euclidean Schwarzschild section has no conical singularities. The associated partition function

$$Z_S^{cl} = e^{-\beta_H M c^2/2} = e^{-E_P^2 \beta_H^2/(16\pi)} \tag{18}$$

gives the desired thermodynamics of black holes, especially the Bekenstein-Hawking entropy

$$S_{BH}/k_B = \frac{A}{4 l_P^2} \ , \ l_P^2 = G\hbar/c^3 \ . \tag{19}$$

The approach appears to become "unconventional", if one includes the term $I_E^{(2)}$ for which the path integral (14) is a Gaussian one which - formally - is proportional to the inverse square root of the product of the eigenvalues of the operator A (=$(\det(A))^{-1/2}$). For the Schwarzschild case the term $I_E^{(2)}$ has been evaluated by Gibbons and Perry[14]. A crucial point is that the operator A has just one negative eigenvalue[4] which contributes a factor i to the partition function. The combined contribution to the partition function Z_S^{scl} of the classical euclidean Schwarzschild solution and the fluctuations

around it is given by [4, 14]

$$\begin{aligned} Z_S^{scl} &= iZ_g Z_{GHP} \ , \\ Z_{GHP} &= \frac{1}{2}\frac{V}{64\pi^3 l_P^3}(a\beta_H)^{212/45} e^{-E_P^2\beta_H^2/(16\pi)} \ , \ Z_g = e^{\pi^2 V/(c^3\hbar^3\beta_H^3)} \end{aligned} \quad (20)$$

where V is the spatial volume of the system and a is a "cutoff" associated with the ζ-function regularization of $\det(A)$. (The index "GHP" stands for "Gibbons, Hawking, Perry".) The factor Z_g is the same as one would obtain for flat space ($\overset{0}{g}_{\mu\nu} = \delta_{\mu\nu}$). It represents the gas of free thermal gravitons surrounding the black hole.

Except for the spatial volume factor the structure of Z_{GHP} in eq. (20) is very similar to that of Z_i in eq. (6) above: If $\sigma^2 = t/(4\pi)$ - see ref. [1] - then the exponentials in both cases are equal for $\beta = \beta_H$. The factors with the powers of β - which in eq. (20) comes from short-distance fluctuations around the classical solution - are different in both cases. However this is not surprising because both approaches are so different. These factors may give rise to logarithmic corrections[1] to the Bekenstein-Hawking entropy (19) which is dominated by the exponential. I shall come back to this in the next chapter.

Hawking has argued[2, ch. 15.8] that the partition function (20) has to be imaginary in order for the density of states $N(E)$ in the Laplace transforms

$$Z(\beta) = \int_0^\infty dE N(E) e^{-\beta E} \ , \quad N(E) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\beta Z(\beta) e^{\beta E} \quad (21)$$

to be real if Z^{cl} from eq. (18) is inserted into the last integral and which exists only if the contour of integration is rotated by $\pi/2$! Because of this Hawking has concluded[2] that the canonical ensemble is inappropriate for the description of black holes.

Recently Hawking and Horowitz have discussed[15] a more refined version of the relationship between the classical euclidean gravitational action (14) - especially its boundary terms- and the entropy of the system. This work is related to that of York and collaborators[16] who suggested to improve the approach of Gibbons and Hawking[13] and Gross, Perry and Yaffe[4] by employing a more microcanonical framework. Related is also the work of Iyer and Wald[17].

In any case, the imaginary parts Z_{GHP} of eq. (20) or Z_i of eq. (6), respectively, do seem to yield interesting though unconventional thermodynamical

properties, if one treates them as if they were "normal" partition functions. As they imply negative heat capacities they signal instabilities of the system. This leads us to a closely related interpretation of the imaginary part of a partition function, namely of being associated with the decay of a metastable state into stable one in the form of nucleation:

4 Imaginary part of partition functions and droplet nucleation

In a series of very influential papers[3] Langer discussed the imaginary part of partition functions for systems with a 1st-order phase transition where the transition from a metastable state to a stable one is initiated by the "nucleation" of droplets. Langer suggested that the nucleation transition rate is proportional to the imaginary part of the partition function. Coleman and Callen[18] discussed this approach in the context of instanton solutions in the euclidean version of otherwise lorentzean field theories and their role for the decay of "false" (metastable) vacua. The methods involved have found wide applications in different fields of physics[19].
The main procedure is very similar to the one described above for the euclidean section of the Schwarzschild solution: The instantons (or "bounces") are classical solutions of the "euclideanized" field equations satisfying appropriate boundary conditions of a certain path integral. The quantum (or thermal) fluctuations around these classical solutions leave the system stable if all the eigenvalues of the second-variation operator A (see above) are positive. However, if one of the eigenvalues is negative, then the classical solution is not related to a minimum of the euclidean action integral, but only to a saddle point and the fluctuations can drive the system from its metastable state into a more stable lower one.
The transition rate is essentially determined by $\Im(Z^{scl})/Z_0$, where Z_0 is the real part of the canonical partition function (which represents the metastable state) and by the absolute value of the negative eigenvalue.
It is this framework in which Gross, Perry and Yaffe[4] interpreted the contribution (20) of the euclidean Schwarzschild solution ("Schwarzschild instanton"):
Eq. (20) represents the "1-instanton" contribution to the partition function.

The next step is a dilute-gas approximation: Neglecting the "interactions" between the instantons, the contribution of N of them to the grand partition function Z_G is

$$Z_g \frac{1}{N!}(iZ_{GHP})^N \ . \tag{22}$$

Summing over all N gives

$$Z_G = Z_g e^{iZ_{GHP}} \tag{23}$$

and the grand canonical potential

$$\Psi(\beta, V, \alpha) = \ln Z_G = \beta p V \ , \quad d\Psi = -U d\beta + p\beta dV - \bar{N} d\alpha \ , \tag{24}$$

($\alpha \equiv -\mu\beta, \mu$: chemical potential, p: pressure) here has the form

$$\Psi = \ln Z_g + iZ_{GHP} \ , \tag{25}$$

whereas the general structure is[3,19]

$$\Psi = \ln Z_0 + i\Im(Z^{scl})/Z_0 \ . \tag{26}$$

Because of the special form of Z^{scl} in (20) the factor Z_g, representing the metastable graviton gas (i.e. $Z_g = Z_0$) drops out in eq. (26)!
According to Langer[3] (and others[18, 19]) the rate Γ of transition per unit volume from the metastable state (here the graviton gas) to the stable state (here the black hole) is given by

$$\Gamma = \frac{|\kappa|}{\pi}\Im(\psi) = \frac{|\kappa|}{\pi}\Im(Z^{scl}/Z_0) \ , \quad \psi = \Psi/V \ \ , \tag{27}$$

were κ is the single negative eigenvalue mentioned above (that a large class of systems with possible negative eigenvalues has just one of them was proven by Coleman[20]).
The factor $|\kappa|$ in eq. (27) is a dynamical one, depending on nonequilibrium properties of the system[3,19]. As to more recent evaluations of $|\kappa|$ for relativistic systems see refs.[21].
The relation between the entropy (19) and the exponential factor $\exp(-\beta^2/(16\pi))$ in the transition rate (27) associated with eq. (25) is perhaps not so obvious anymore. However, in the case of nucleation one writes[3,19]

$$\Im(Z^{scl})/Z_0 = e^{-\beta(F^{scl} - F_0)} \ , \ F^{scl} = -\frac{1}{\beta}\ln\Im(Z^{scl}) \ , \ F_0 = -\frac{1}{\beta}\ln Z_0 \tag{28}$$

and interprets $\hat{F} = F^{scl} - F_0$ as the excess free energy of the critically large droplet. In our case we have $Z_0 = Z_g$ and therefore $\hat{F}_S = -(1/\beta)\ln Z_{GHP}$. In this sense Z_{GHP} represents the partition function of the *bare* black hole, i.e. it describes the thermodynamics of the black hole without the surrounding graviton gas.

Thus, the more sophisticated nucleation picture essentially leads to the same thermodynamical properties of Schwarzschild black holes as the "naive" use[1] of the imaginary part of the partition function!

Still, it seems desirable to have a more systematic analysis of the relationship between the two approaches.

Page[22] has given a detailed analysis of the rate (27) for the transition of a gas into a black hole. The picture that black holes are formed by nucleation from a gas was already discussed by Gibbons and Perry[23].

5 Schwarzschild black hole quantum statistics and Ising droplet nucleation in 2 dimensions

One of the most amazing parts of the whole analysis presented here is that the quantum partition function (3) associated with a Schwarzschild black hole is the same as that of the (primitive) Ising droplet model for nucleation (in 1st-order phase transitions) in 2 dimensions[24]:

Suppose a d-dimensional ($d \geq 2$) lattice of N Ising spins below the critical temperature T_c to be in a positive magnetic field H so that almost all the spins are up (+1). If one then slowly turns the magnetic field negative, the system comes into a metastable state in which the total magnetization is still positive, before finally becoming negative, too. A simple model to describe the transition into that stable state is the following: Assume that (small) droplets containing l down-spins form, in a background of up-spins. The droplets are supposed to be noninteracting (dilute-gas approximation) and the number n_l of droplets of size l to be given by

$$n_l = N e^{-\beta \epsilon_l} \; , l = 1, \dots \; . \tag{29}$$

The droplet formation energy ϵ_l is assumed to consist of two (competing) terms, the bulk energy $2Hl$ and a surface energy term $\phi l^{(d-1)/d}$, so that

$$\epsilon_l = 2Hl + \phi l^{(d-1)/d} \ . \tag{30}$$

The interesting part of the canonical partition function Z_N is the finite sum

$$Z_N/N = \sum_l e^{-\beta 2Hl - \beta\phi l^{(d-1)/d}} \ . \tag{31}$$

Letting N go to ∞ and employing the grand canonical ensemble in exactly the same way as for the instanton gas above yields the following grand canonical potential ψ per spin

$$\psi(\beta, H) = \sum_{l=0}^{\infty} e^{-2lH\beta} e^{-l^{(d-1)/d}\phi\beta} \ . \tag{32}$$

The theoretical investigations of the function ψ have focussed mainly on its behaviour for negative H (where the sum no longer converges!) in the neighbourhood of $H = 0$. It is clear that ψ must have a singularity at $H = 0$, the so-called "condensation point". Many approximate analytical calculations and rigorous estimates[25] suggested that ψ has an essential singularity there, a supposition supported by numerical calculations[26].
Now for $d = 2$ the series (32) is just the same as (3) which has been summed exactly in ref.[1] by using Lerch's observation[27] that the relation

$$\begin{aligned} e^{-\sqrt{nx^2}} &= \frac{|x|}{\sqrt{\pi}} \int_0^\infty dv e^{-x^2v^2/4 - n/v^2} \\ &= \frac{|x|}{2\sqrt{\pi}} \int_0^\infty \frac{d\tau}{\tau^{3/2}} e^{-x^2/(4\tau) - n\tau} \end{aligned} \tag{33}$$

turns the series into a geometrical one which can be summed under the integral sign and then continued analytically. We only have to put $t = -2\beta H, x = \phi\beta$ and then get from eq. (6) the *exact* result

$$\Im(\psi)(\beta, H < 0) = \frac{\sqrt{\pi}}{4\sqrt{2}} \phi\beta^{-1/2}|H|^{-3/2} e^{-(\phi^2\beta)/(8|H|)} \ . \tag{34}$$

Thus, there is indeed an essential singularity at $H = 0$. Instead of $|H|^{-3/2}$ in front of the exponential factor the approximation methods of Langer[3]

and Günther et al.[25] yield the factor $|H|$. The modified method used by Harris[26] gives the correct $|H|$-dependence!
The real part of ψ is to be calculated by means of the principal value (dispersion) integral (7) (if the sum (32) starts with $l = 1$ etc. corresponding changes have to be made[1]).
In ref.[1] I pointed out that the real part (7) can become negative for small x. The droplet nucleation picture might help to understand the (physical?) background of this.
For a recent discussion of metastability in the 2-dimensional Ising model itself see ref.[28].
It was already stressed in ref.[1] that the series (3) obeys the heat equation $\partial_t Z = \partial_x^2 Z$. The corresponding partial differential equation for ψ in d=3 dimensions is $\partial_t^2 \psi = -\partial_x^3 \psi$. This ought to help analysing the behaviour of ψ in the neighbourhood of $t = 0$ for d=3, too! I shall only indicate what to expect: As that partial differential equation is invariant under the scale transformation $t \to \lambda t, x \to \lambda^{2/3} x$, there should exist solutions of the type $f(y), y = x^3/t^2$. Inserting this gives for $f(y)$ the ordinary diff. eq. $27f''' + (4 + 54/y)f'' + 6(1/y + 1/y^2)f' = 0$. For $t \to 0, y \to \infty$, there is the approximate solution $f'' \sim \exp(-(4/27)y)$ which agrees with the form of the expected essential singularity as to small negative H in 3 dimensions[25]. However, more analysis is certainly necessary here.

6 Conclusions

It appears that there can be hardly any doubt that the square root spectrum (1) is closely related to the expected thermodynamics of black holes. It is therefore important to understand its physical meaning and the theoretical background better. Because it implies that the area A of the horizon is proportional to $\hbar$ one should probably look here for an interpretation[11]. This corresponds to other recent investigations which stress the importance of bifurcate horizons and the associated surfaces for understanding the thermodynamics of black holes[29].
Furthermore, the discussion above shows that a "naive" and a droplet nucleation interpretation, respectively, of the imaginary part of a partition function provide just two different aspects of the thermodynamical properties of metastable systems.

Similarly important is the physical understanding of the degeneracy (2) which is equally essential for the derivation of the thermodynamics. In the Ising droplet model above g is replaced by the factor $\exp -\beta H$ which represents the influence of the *driving* external (negative) magnetic field. The same role can have a (positive) chemical potential[3,19]. Bekenstein and Mukhanov[11] used information theoretical arguments to put $g = 2$. However, the arguments above go through with any $g > 1$. The physical background of the degeneracies (2) probably needs further understanding. (The "driving field" behind $g > 1$ is most likely the gravitational attraction which leads to the black hole.)
In addition it will be interesting to see whether and how the above considerations can be extended to the thermodynamics of the Reissner-Nordstrøm model[30] where investigations similar to those of refs.[8,9] for the Schwarzschild case exist[31].
Very exciting is the possibility to map the quantum statistics of a Schwarzschild black hole onto the statistics of the 2-dimenionsal Ising model for droplet nucleation. This might allow to learn from a known field of physics for an unknown one.
Finally it is surprising that the pursuit of black hole physics leads to an exact mathematical solution for an old condensed-matter problem.

I am very much indebted to Malcolm Perry for drawing my attention to refs. [3] and [4].
Note added in response to a question of the referee:
If one considers a Schwarzschild black hole in d+1 space-time dimensions[32], $d \geq 3$, then its Schwarzschild radius R_S is proportional to $M^{1/(d-2)}$, where M is the mass of the system. Thus, if we again assume the time interval Δ in eq. (12) above to be proportional to R_S, then $\sqrt{n}$ in eq. (13) is replaced by $n^{(d-2)/(d-1)}$ and the resulting partition function is the same as that of the Ising droplet model for nucleation in d-1 dimenions (see eq. (32))!

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